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The solution of a problem in contact fracture mechanics on the nucleation and development of a bridged crack in the hub of a friction pair☆

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Abstract

The contact deformation of the hub of a plunger pair is considered. It is assumed that, during the repeated reciprocating motion of the plunger, a crack is initiated and fracture of the materials of the elements of the contact pair occurs. The problem of the equilibrium of the hub of a friction pair with a crack nucleus reduces to solving a system of non-linear singular integrodifferential equations with a Cauchy-type kernel. The normal and shear forces in the zone where the crack originates are found from the solution of this system of equations. The condition for the appearance of a crack is formulated, taking account of the criterion of the limit traction of the bonds in the material. A problem for the plunger of a friction pair as applied to a borehole sucker rod pump is considered as an example. In conclusion, the case when there are several arbitrarily distributed rectilinear bridged cracks, with bonds between the crack faces in the end zone, close to the contact surface of the hub is investigated.

1. Formulation of the problem

Experience in using a plunger pair shows that the initiation of cracks and fracture of the materials of the components of the contact pair occur during repeated reciprocating motion. It is therefore necessary in the planning stage of the construction of sliding pairs to take account of the possibility of the occurrence of cracks and to carry out a limit analysis of the components of the contact pair. In view of known observations of the physical fracture process, the following natural sequence is adopted when analysing the initiation and development of a bridged crack in the hub of a friction pair: a) investigation of the occurrence in the hub of a pre-fracture zone and the formation of a crack with bonds between its faces in the end domains of arbitrary size, b) the analysis of the development of a bridged crack. Pre-fracture zones will arise in proportion to the loading of the hub during the operation of the friction pair with force and thermal loads, and these zones are modelled as domains in which the interparticle bonds in the material have been weakened. The interaction of the surfaces of these domains is modelled by the introduction of a pre-fracture zone of the bonds, which have a specified deformation pattern, between the crack faces. The physical nature of these and the dimensions of the pre-fracture zones depend on the form of the material. Since the above-mentioned zones (layers) are small compared with the remaining part of the hub, they can be conceptually eliminated by replacing them with cuts,

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the surfaces of which interact with one another according to a certain law that corresponds to the action of the material which has been removed. Taking account of these effects in fracture mechanics is an important but exceedingly difficult problem.

In the case being investigated, the occurrence of a crack nucleus involves the transition of a pre-fracture zone into a domain where there are ruptured bonds between the surfaces of the material. Here, the size of the pre-fracture zone is unknown in advance and has to be determined when solving the problem.

Investigations[1–5] of the occurrence of domains with a disrupted structure of the material show that, during the initial stage, the pre-fracture zones have the form of a narrow elongated layer and then, when the load is increased, a secondary system of zones suddenly appears, and these zones contain material with partially ruptured bonds.

For a mathematical description of the formation of a defect such as a crack in the hub of a contact pair when operating, we arrive, in the case being considered, at a contact problem involving the indentation of a plunger into the inner surface of a hub when there is a pre-fracture zone in the hub. The zone is orientated in the direction of the maximum tensile stresses arising in the hub. The origin of the local system of coordinates $x_1O_1y_1$ is located at the centre of the zone, and the x_1 axis coincides with the axis of the zone and makes an angle α_1 to the x axis (Fig. 1).

The surfaces of the pre-fracture zone (SPFZs) interact in such a way that this interaction (the bonds between the surfaces) restrains the formation of a defect (crack).

For a mathematical description of the interaction of the SPFZs, it is assumed that there are bonds between them for which the law of deformation is known. Under the action of external loads on the hub, normal $q_{y_1}(x_1)$ and shear $q_{x_1y_1}(x_1)$ tractions will arise in the bonds joining the SPFZs. Consequently, normal and shear stresses, which are numerically equal to $q_{y_1}(x_1)$ and $q_{x_1y_1}(x_1)$ respectively, will be applied to the SPFZs. The magnitudes of these stresses are unknown in advance and are to be determined when solving of a boundary-value problem in contact fracture mechanics.

The stress-strain state of the hub of the contact pair is considered. During the operation of the "hub - plunger" pair, there is a force interaction between the contact surfaces of the hub and the plunger, and friction forces arise which lead to wear of the material of the pair. To determine the contact pressure, it is necessary to consider[6] the contact problem of the indentation of the plunger into the hub surface when there is wear.

Suppose a plunger with a radius of the surface R_0' and mechanical characteristics G_1 and μ_1 is pressed onto a certain segment, unknown in advance, of the inner surface of radius R_0 of a hub with mechanical characteristics G and μ. It is assumed that the hub with an outer radius R_1 is clamped by means of a rigid yoke. It is assumed that the conditions of plane strain are satisfied.

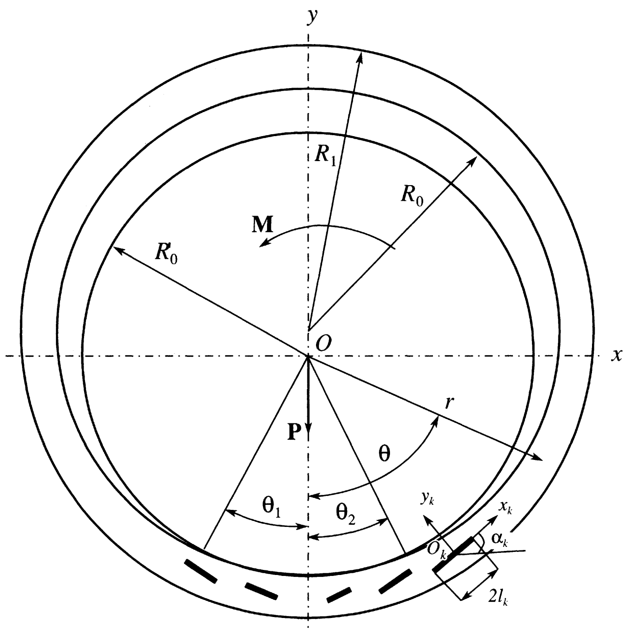

Fig. 1.

The condition relating the displacements of the hub and the plunger is written in the form[6,7]

$$v_1 + v_2 = \delta(\theta), \quad \theta_1 \le \theta \le \theta_2 \tag{1.1}$$

Here $\delta(\theta)$ is the sag of the points of the surface of the hub and the plunger, which is determined by the form of the inner surface of the hub and the plunger surface and, also, by the magnitude of the pressing force **P**; $\theta_2 - \theta_1$ is the magnitude of the contact angle contact (area).

The pressure $p(\theta, t)$ is distributed asymmetrically over the contact area and creates a moment **M** about to the centre of the plunger.

In the contact area, apart from the normal pressure, there is a shear stress $\tau_{r\theta}$ which is related to the contact pressure $p(\theta, t)$ by the Amonton - Coulomb law

$$\tau_{r\theta}(\theta, t) = fp(\theta, t) \tag{1.2}$$

where f is the coefficient of friction of the “hub - plunger” pair.

The shear forces (friction forces) $\tau_{r\theta}(\theta, t)$ release heat in the contact area. The total amount of heat per unit time is proportional to the power of the friction forces, and the amount of heat released at the point in the contact area with coordinate θ will be equal to

$$Q(\theta, t) = Vfp(\theta, t) \tag{1.3}$$

where V is the mean rate of the plunger displacement with respect to the hub over a period.

The overall amount of heat $Q(\theta, t)$ will be equal to the sum of the amount of heat $Q_b(\theta, t)$, which leads to heating of the hub, and the amount of heat $Q_1(\theta, t)$ which leads to heating of the plunger, that is,

$$Q = Q_b + Q_1$$

For the radial displacement of the hub, we shall have

$$v_1 = v_{1y} + v_{1u} \tag{1.4}$$

Here v_1 are the radial thermoelastic displacements of the points of the contact surface of the hub (CSH) and v_{1u} are the displacements caused by wear of the hub surface.

To simplify the problem, we will neglect the displacements caused by crumpling of the microprotuberances on the hub surface. An analysis of the contact problem, taking account of the roughness of the components of the contact pair, has been carried out previously.[8,9]

The relation for the radial displacement v_2 of the plunger can be written in a similar manner to relation (1.4).

The wear on the components of a friction pair in oil production equipment is assumed to be due to abrasion. The rate of change of the displacement of the surface accompanying the hub wear will be[6]

$$\frac{dv_{1u}}{dt} = K_b p(\theta, t) \tag{1.5}$$

where K_b is the abrasion coefficient of the hub material.

Heating of the bush occurs as a result of friction during the reciprocating motion of the plunger. Since the frequency of the motion of the plunger is quite high, we will treat the problem as a stationary problem. In order to determine the displacements v_{ly}, it is necessary to solve the following thermoelasticity problem for a hub with a pre-fracture zone

$$\Delta T = 0 \text{ in the hub} \tag{1.6}$$

$$r = R_0{:}\ \lambda\frac{\partial T}{\partial r} = -Q_b \text{ on the CSH; } \lambda\frac{\partial T}{\partial r} + \alpha_1(T - T_c) = 0 \text{ outside the CSH} \tag{1.7}$$

$$r = R_1{:}\ \lambda\frac{\partial T}{\partial r} + \alpha_2(T - T_c) = 0 \tag{1.8}$$

$$r = R_0{:}\ \sigma_r = -p(\theta, t), \quad \tau_{r\theta} = -fp(\theta, t) \text{ on the CSH; } \quad \sigma_r = 0, \quad \tau_{r\theta} = 0 \text{ outsid the CSH} \tag{1.9}$$

$$r = R_1{:}\ u_r = 0, \quad u_\theta = 0 \tag{1.10}$$

$$\sigma_n = q_{y_1}(x_1), \quad \tau_{nt} = q_{x_1y_1}(x_1) \text{ on the SPFZs} \tag{1.11}$$

Here λ is the coefficient of heat conduction of the hub, α_1 is the coefficient of heat transfer from the inner surface of the hub, α_2 is the heat exchange coefficient of the outer surface of the cylinder with the environment at a temperature T_c, n and t are the normal and the tangent to the contour of the pre-fracture zone, u_r is the radial component and u_θ the tangential component of the displacement vector of the points of the contour of the hub, and σ_r, σ_θ, $\tau_{r\theta}$ are the components of the stress tensor.

The quantity Q_b depends on the contact pressure and is proportional to the overall amount of heat released $Q(\theta, t)$; the coefficient of proportionality is determined experimentally.

The thermoelasticity problem of determining the displacements of the contact surface of the plunger is formulated in a similar manner.

The magnitudes of θ_1 and θ_2, that is, of the ends of the segment over which the plunger and the hub are in contact, are unknown. In order to determine them, we will use a condition which expresses the continuous fall of the pressure $p(\theta)$ to zero when the point θ falls outside the segment where the surfaces touch

$$p(\theta_1) = 0, \quad p(\theta_2) = 0 \tag{1.12}$$

The equations and conditions (1.1)–(1.12) have to be supplemented with a relation between the expansion of the pre-fracture zone and the bond tractions. Without loss of generality, we will represent this relation in the form[5]

$$(v^+ - v^-) - i(u^+ - u^-) = C(x_1, \sigma)[q_{y_1}(x_1) - iq_{x_1y_1}(x_1)], \quad \sigma = \sqrt{q_{y_1}^2 + q_{x_1y_1}^2} \tag{1.13}$$

The function $C(x_1, \sigma)$ can be regarded as the effective compliance of the bonds which depends on their tension, σ is the modulus of the vector of the bond tractions, and $(v^+ - v^-)$ is the normal and $(u^+ - u^-)$ is the tangential component of the expansion of the pre-fracture zone.

In order to determine the value of the external load (the contact pressure) at which a crack is initiated, it is necessary to supplement the formulation of the problem with a condition (criterion) for the appearance of a crack (the rupture of the interparticle bonds in the material). As such a condition, we will adopt the criterion for the critical expansion of the pre-fracture zone

$$|(v^+ - v^-) - i(u^+ - u^-)| = \delta_{cr} \tag{1.14}$$

where δ_{cr} is a characteristic of the fracture toughness of the hub material.

This additional condition enables us to determine the parameters of the contact pair for which a crack appears in the hub.

2. The case of a single pre-fracture zone

In order to solve the problem formulated above, simultaneous solution of the problem of contact wear and the problem of mechanical fracture is necessary.

The solution of the boundary-value problem in the theory of heat conduction is sought by the method of separation of variables. We find the distribution of the excess temperature of the hub $t_b = T - T_c$ in the form

$$t_b = C_1 + C_2 \ln r + \sum_{k=1}^{\infty} (C_1^{(k)} r^k + C_2^{(k)} r^{-k}) \cos k\theta + \sum_{k=1}^{\infty} (A_1^{(k)} r^k + A_2^{(k)} r^{-k}) \sin k\theta \tag{2.1}$$

The constants $C_1, C_2, C_1^{(k)}, C_2^{(k)}, A_1^{(k)}, A_2^{(k)}$ are determined from the boundary conditions of the heat-conduction problem (1.7), (1.8). Because of their length, the corresponding formulae are not presented here.

To solve the thermoelasticity problem, we will use the thermoelastic displacement potential.[10] In the problem being considered, the thermoelastic displacement potential for the hub F is found by solving the differential equation

$$\Delta F = \frac{1 + \mu}{1 - \mu} \alpha t_b \tag{2.2}$$

Here α is the coefficient of linear thermal expansion.

We will seek a solution of Eq. (2.2) in the form

$$F = \sum_{n=0}^{\infty}(f_n\cos\theta + f_n^*\sin n\theta) \tag{2.3}$$

Ordinary differential equations are obtained for the functions $f_n(r)$ and $f_n^*(r)$, the solutions of which are found by the method of variation of the constants. After determining the thermoelastic displacement potential for the hub using well-known formulae,[10] we calculate the stresses $\bar{\sigma}_r, \bar{\sigma}_\theta, \bar{\tau}_{r\theta}$ and displacements $\bar{u}_r, \bar{u}_\theta$ for the hub corresponding to the thermoelastic displacement potential. The stresses and displacements obtained will not satisfy boundary conditions (1.9)–(1.11). For the hub, it is necessary to find a second stress-strain state: $\bar{\bar{\sigma}}_r, \bar{\bar{\sigma}}_\theta, \bar{\bar{\tau}}_{r\theta}, \bar{\bar{u}}_r, \bar{\bar{u}}_\theta$ such that boundary conditions (1.9)–(1.11) are satisfied.

Consequently, to determine the second stress-strain state in the hub, we have the boundary conditions

$$\begin{aligned} &r = R_0\colon \bar{\bar{\sigma}}_r = -p(\theta) - \bar{\sigma}_r, \quad \bar{\bar{\tau}}_{r\theta} = -fp(\theta) - \bar{\tau}_{r\theta} \text{ on the CSH} \\ &\bar{\bar{\sigma}}_r = -\bar{\sigma}_r, \quad \bar{\bar{\tau}}_{r\theta} = -\bar{\tau}_{r\theta} \text{ outside the CSH} \end{aligned} \tag{2.4}$$

$$r = R_1\colon \bar{\bar{u}}_r = -\bar{u}_r, \quad \bar{\bar{u}}_\theta = -\bar{u}_\theta \tag{2.5}$$

$$\bar{\bar{\sigma}}_{y_1} = q_{y_1} - \bar{\sigma}_{y_1}, \ \bar{\bar{\tau}}_{x_1y_1} = q_{x_1y_1} - \bar{\tau}_{x_1y_1} \text{ on the SPFZs} \tag{2.6}$$

Using the Kolosov - Muskhelishvili formulae,[11] boundary conditions (2.4)–(2.6) can be written in the form of a boundary-value problem for finding the complex potentials $\Phi(z)$ and $\Psi(z)$ for the hub.

We will seek the complex potentials in the form[11,12]

$$\Phi(z) = \Phi_1(z) + \Phi_2(z) + \Phi_3(z), \quad \Psi(z) = \Psi_1(z) + \Psi_2(z) + \Psi_3 \tag{2.7}$$

$$\Phi_1(z) = \sum_{k=-\infty}^{\infty} a_k z^k, \quad \Psi_1(z) = \sum_{k=-\infty}^{\infty} b_k z^k \tag{2.8}$$

$$\Phi_2(z) = \frac{1}{2\pi}\int_{-l_1}^{l_1}\frac{g_1(t)}{t-z_1}dt, \quad \Psi_2(z) = \frac{1}{2\pi}e^{-2i\alpha_1}\int_{-l_1}^{l_1}\left[\frac{\overline{g_1(t)}}{t-z_1} - \frac{\bar{T}_1}{(t-z_1)^2}e^{i\alpha_1}g_1(t)\right]dt \tag{2.9}$$

$$\Phi_3(z) = \frac{1}{2\pi}\int_{-l_1}^{l_1}\left[-\frac{1}{zT_2}e^{i\alpha_1}g_1(t) + \frac{1-T_1\bar{T}_1}{\bar{T}_1T_2^2}e^{-i\alpha_1}\overline{g_1(t)}\right]dt$$

$$\begin{aligned} \Psi_3(z) = \frac{1}{2\pi z}\int_{-l_1}^{l_1}\Bigg\{&\left[\frac{1}{zT_1} - \frac{1}{z^2} - \frac{1}{z^2T_2} + \frac{\bar{T}_1^2}{T_2^2}\right]e^{i\alpha_1}g_1(t) + \\ &+\left[\frac{1-T_1\bar{T}_1}{z\bar{T}_1T_2^2} - \frac{1}{1-zT_1} - \frac{2(1-T_1\bar{T}_1)}{T_2^3}\right]e^{-i\alpha_1}\overline{g_1(t)}\Bigg\}dt \end{aligned} \tag{2.10}$$

Here,

$$T_1 = te^{i\alpha_1} + z_1^0, \quad z_1 = e^{-i\alpha_1}(z - z_1^0), \quad T_2 = 1 - z\bar{T}_1$$

and $g_1(x_1)$ is the required function, which characterizes the expansion of the pre-fracture zone:

$$g_1(x) = \frac{2G}{i(1+k_0)}\frac{\partial}{\partial x}[u_1^+(x,0) - u_1^-(x,0) + i(v_1^+(x,0) - v_1^-(x,0))], \quad k_0 = 3 - 4\mu \tag{2.11}$$

By requiring that complex potentials (2.7)–(2.10) should satisfy the original boundary conditions (2.4) and (2.5), we obtain an infinite system of equations in the coefficients a_k and b_k. The contact pressure expansion coefficients

$$p(\theta, t) = \alpha_0 + \sum_{k=1}^{\infty} (\alpha_k \cos k\theta + \beta_k \sin k\theta) \tag{2.12}$$

and, also, integrals of the required function $g_1(t)$ occur on the right-hand sides of these systems. The solution of these systems, that is, the expression of a_k and b_k in terms of the quantities α_k and β_k and integrals of the function $g_1(t)$, does not present any difficulty (see Ref. 11, § 59).

On satisfying the boundary condition (2.6) on the SPFZs by means of functions (2.7)–(2.10), we obtain the following singular integral equation in the unknown function $g_1(x_1)$

$$\int_{-l_1}^{l_1} [R_{11}(t, x_1)g_1(t) + S_{11}(t, x_1)\overline{g_1(t)}]dt = \pi f_1(x_1), \quad |x_1| \le l_1 \tag{2.13}$$

where

$$\begin{aligned} f_1(x_1) &= f_1^0 + q_{y_1} - iq_{x_1y_1} - (\bar{\sigma}_{y_1} - i\bar{\tau}_{x_1y_1}), \\ f_1^0(x_1) &= -[\Phi_1(x_1) + \overline{\Phi_1(x_1)} + x_1\overline{\Phi_1'(x_1)} + \overline{\Psi_1(x_1)}] \end{aligned} \tag{2.14}$$

The variables x_1, t and l_1, which are dimensionless quantities having been divided by R_0; R_{nk} and S_{nk}, are determined using known relations (Ref. 12, formula (VI. 61)).

It is necessary to add the condition for the uniqueness of the displacement on passing around the contour of the pre-fracture zone

$$\int_{-l_1}^{l_1} g_1(t)dt = 0 \tag{2.15}$$

to the singular integral equation for the interior pre-fracture zone.

The radial displacement v_1 of the contact surface of the hub is found using complex potentials (2.7)–(2.10), the Kolosov - Muskhelishvili formulae and by integrating the kinetic equation for the wear of the hub material.

The thermoelasticity problem for the plunger is treated in a similar way. The radial displacement v_2 of the contact surface of the plunger is found using the solution of the thermoelasticity problem for the plunger and the kinetic equation for the wear of the plunger material. The quantities v_1 and v_2 which are found are substituted into the main contact Eq. (1.1). In order to cast the main contact equation in an algebraic form, the unknown contact pressure functions are sought in the form of the expansions

$$\begin{aligned} p(\theta, t) &= p_0(\theta) + tp_1(\theta) + \ldots; \\ p_s(\theta) &= \alpha_0^s + \sum_{k=1}^{\infty} (\alpha_k^s \cos k\theta + \beta_k^s \sin k\theta), \quad s = 0, 1, \ldots \end{aligned}$$

In order to construct an algebraic system with the aim of finding the required coefficients α_k and β_k, we compare coefficients of like trigonometric functions. As a result, we obtain an infinite algebraic system in α_k^0 $(k = 0, 1, 2, \ldots)$, β_k^0 $(k = 1, 2, \ldots)$ and α_k^1, β_k^1, etc. This system of equations turns out to be non-linear due to the unknown quantities θ_1 and θ_2.

Using the procedure for converting to an algebraic form (see Ref. 13, Appendix), the singular integral Eq. (2.13) with condition (2.15) reduces to a system of M algebraic equations for determining the M unknowns $g_1(t_m)$ ($m = 1, 2, \ldots, M$).

$$\frac{1}{M}\sum_{k=1}^{M} l_1[g_1(t_m)R_{11}(l_1t_m, l_1x_r) + \overline{g_1(t_m)}S_{11}(l_1t_m, l_1x_r)] = f_1(x_r), \quad r = 1, 2, \ldots, M-1$$
$$\sum_{m=1}^{M} g_1(t_m) = 0 \tag{2.16}$$

where

$$t_m = \cos\frac{2m-1}{2M}\pi, \quad m = 1, 2, \ldots, M, \quad x_r = \cos\frac{\pi r}{M}, \quad r = 1, 2, \ldots, M-1$$

If we change to complex-conjugate quantities in system (2.16), a further M algebraic equations are obtained.

The unknown values of the normal force $q_{y_1}(x_r)$ and the shear force $q_{x_1y_1}(x_r)$ at the mesh points of the pre-fracture zone occur on the right-hand sides of system (2.16).

The additional relation (1.13) is the condition which determines the unknown bond tractions between the surfaces of the pre-fracture zone. In the problem being considered, it is more convenient to write this additional condition for an arbitrary expansion of the pre-fracture zone.

$$\frac{\partial}{\partial x_1}[\upsilon^+(x_1, 0) - \upsilon^-(x_1, 0) - i(u^+(x_1, 0) - u^-(x_1, 0))] =$$
$$= \frac{\partial}{\partial x_1}[C(x_1, \sigma)(q_{y_1}(x_1) - iq_{x_1y_1}(x_1))] \tag{2.17}$$

Using the solution obtained, we can write

$$g_1(x) = \frac{2G}{1+k_0}\frac{\partial}{\partial x_1}[C(x_1, \sigma)(q_{y_1}(x_1) - iq_{x_1y_1}(x_1))] \tag{2.18}$$

where x_1 is the affix of the points of the SPFZs. This complex equations serves for determining the bond tractions q_{y_1} and $q_{x_1y_1}$ between the SPFZs.

In order to construct the missing algebraic equations for determining approximate traction values $q_{y_1}(t_m)$ and $q_{x_1y_1}(t_m)$ at the mesh points, we require that conditions (2.18) are satisfied at the mesh points t_m ($m = 1, 2, \ldots, M$) contained within the pre-fracture zone. The finite difference method is used here.

As a result, a complex algebraic system of M equations is obtained for determining the approximate values of $q_{y_1}(t_m)$, $q_{x_1y_1}(t_m)(m = 1, 2, \ldots, M)$ at the mesh points of the pre-fracture zone. Here, account has been taken of the boundary conditions

$$t_0 = \pm 1: q_{y_1}(l_1t_0) = 0, \quad q_{x_1y_1}(l_1t_0) = 0$$

which correspond to the conditions

$$\upsilon^+(\pm l_1, 0) - \upsilon^-(\pm l_1, 0) = 0, \quad u^+(\pm l_1, 0) - u^-(\pm l_1, 0) = 0$$

The two complex equations determining the dimensions of this zone are not sufficient to complete the algebraic equations which have been obtained.

Since the solution of integral Eq. (2.13) is sought in the class of functions (stresses) which are bounded everywhere, then, to system (2.16), it is necessary to add the conditions for the stresses at the ends of the zone $x_1 = \pm l_1$ to be bounded. These conditions have the form

$$\sum_{m=1}^{M}(-1)^{M+m}g_1(t_m)\operatorname{tg}\frac{2m-1}{4M}\pi = 0, \quad \sum_{m=1}^{M}(-1)^{m}g_1(t_m)\operatorname{ctg}\frac{2m-1}{4M}\pi = 0 \tag{2.19}$$

On account of the unknown size of the zone, the resulting algebraic system (2.16)–(2.19) is non-linear.

The systems of equations in a_k, b_k, α_k, β_k, $g_1(t_m)$, $q_{y_1}(t_m)$, $q_{x_1y_1}(t_m)\,(m = 1, 2, \ldots, M)$ obtained enable us, in the case of a specified external load, to find the stress-strain state of the hub when there is a pre-fracture zone in the hub, the contact pressure, the temperature distribution and, also, the abrasive wear of the components of the contact pair.

The simultaneous solution of the resulting systems of equations enables us to find approximate values of the coefficients a_k, b_k, α_k, β_k, the values of the functions $v(t_m)$, $u(t_m)$, $q_{y_1}(t_m)$, $q_{x_1y_1}(t_m)\,(m = 1, 2, \ldots, M)$ and the dimensions of the pre-fracture zone.

On account of the unknown quantities θ_1, θ_2, l_1, the joint system of equations is non-linear even in the case of linear elastic bonds. The method of successive approximations is used to solve it, the essence of which is as follows. We solve the joint algebraic system for certain specific values of θ_{1*} and θ_{2*} for the remaining unknowns (which have been enumerated above). The remaining unknowns occur linearly in the joint system. The values of θ_{1*}, θ_{2*}, l_{1*} and the values of the remaining unknowns corresponding to them will not, generally speaking, satisfy conditions (1.12) and (2.19). Hence, choosing the values of θ_1, θ_2, l_1, we repeat the calculations many times until Eqs. (1.12) and (2.19) are satisfied with a specified accuracy.

In each approximation, the joint algebraic system was solved by Gauss method with a choice of the principal element.

In the case of a non-linear deformation of the bonds, an iterative method is used to determine the tractions in the pre-fracture zone which is similar to the method of elastic solutions.[14] It is assumed that the law of deformation of the interparticle bonds in the zone is linear when $V = \sqrt{u^2 + v^2} \le V_*$. The first step in the iterative calculation procedure consists of solving the system of resolving equations for the linear elastic bonds. Subsequent iterations are only carried out when the inequality $V(x) > V_*$ holds in a part of the zone. In the case of such iterations, the system of resolving equations is solved for quasi-elastic bonds (cohesive forces) with an effective compliance which is variable along the zone and depends on the magnitude of the modulus of the traction vector in the bonds, obtained in the preceding step of the calculation. The effective compliance is calculated in a similar manner to that used to find the secant modulus in the method of variable elastic parameters.[15] It is assumed that the successive approximation process is terminated only when the tractions in the zone obtained in two successive steps differ by a small amount.

The non-linear part of the curve for the deformation of the bonds was represented in the form of a bilinear relation,[4,5] the rising part of which corresponded to the elastic deformation of the bonds $(0 < V(x_1) < V_*)$ with maximum tension of the bonds. When $V(x_1) > V_*$, the deformation was described by a non-linear relation which is determined by the two points (V_*, σ_*) and $(\delta_{cr}, \sigma_{cr})$ while, when $(\delta_{cr} \ge \sigma_*)$, we have an increasing linear relation (linear strengthening corresponding to the elastoplastic deformation of the bonds).

The distribution of the normal tractions q_{y_1}/p_0 in the pre-fracture zone is shown in Fig. 2. The compliance of the bonds in the normal and tangential directions were assumed to be equal and constant along the zone. The law for the change in the shear tractions along the zone is similar to the change in the normal tractions with the difference that the absolute values of the shear tractions are substantially less and the maximum values of the shear tractions are reached

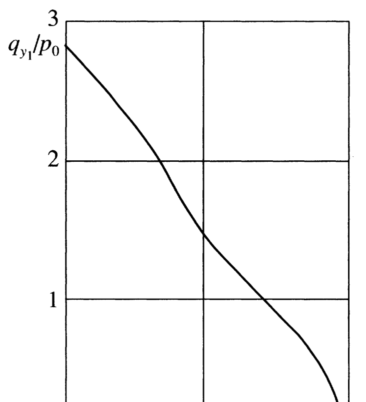

Fig. 2.

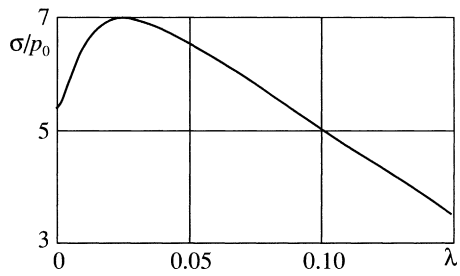

Fig. 3.

in the case of a zone with smaller dimensions. A graph of the distribution of the maximum value of the modulus of the vector is shown in Fig. 3 as a function of the relative size $\lambda = l_1/(R_1 - R_0)$ of the zone. The calculations were carried out for a hub as applied to bore hole sucker rod pumps with a speed of 0.2 m/s. The following parameters were taken as the constants

$$2R_0 = 43.12 \text{ mm}, \; 2R_1 = 57 \text{ mm}, \; 2R_0' = 43 \text{ mm}, \tag{2.20}$$

$$f = 0.2, \; E = 1.8 \cdot 10^5 \text{ MPa}, \; \mu = 0.25, \; V_* = 10^{-6} \text{m},$$

$$\sigma_* = 75 \text{ MPa}, \; \sigma_{cr}/\sigma_* = 2, \; \delta_{cr} = 2.5 \cdot 10^{-6} \text{ m}, \; K_b = 2 \cdot 10^{-8}, \tag{2.21}$$

$$C_B = 2 \cdot 10^{-7} \text{ m/MPa}$$

(C_B is the effective compliance of the bonds).

Using the solution of the problem, we now calculate the displacements on the SPFZs.

$$-\frac{1+k_0}{2G}\int_{-l_1}^{x_1} g_1(x_1)dx_1 = \upsilon(x_1, 0) - iu(x_1, 0)$$

Putting $x_1 = x_0$, using a change of variable and replacing the integral with a sum, we find

$$-\frac{1+k_0}{2G}\frac{\pi l_1}{M}\sum_{m=1}^{M_1} g_1(t_m) = \upsilon(x_0, 0) - iu(x_0, 0) \tag{2.22}$$

Here, M_1 is the number of mesh points in the interval $(-l_1, x_0)$. Taking account of the fact that $g_1(t_m) = \upsilon^0(t_m) - iu^0(t_m)$, we find $\upsilon(x_0, 0)$, $u(x_0, 0)$ and the modulus of the vector of the displacements on the SPFZs when $x_1 = x_0$ from relation (2.22)

$$V_0 = \sqrt{u^2 + \upsilon^2} = \frac{1+k_0}{2G}\frac{\pi l_1}{M}\sqrt{A^2 + B^2}; \quad A = \sum_{m=1}^{M_1} \upsilon^0(t_m), \quad B = \sum_{m=1}^{M_1} u^0(t_m) \tag{2.23}$$

The critical condition (1.14) is used to determine the limit state at which a crack occurs. The equality

$$V_0 = \delta_{cr} \tag{2.24}$$

in conjunction with expression (2.23) will then be the condition which determines the limit value of the external load (the contact pressure).

Taking account of relation (1.13), the limit condition can also be written in the form

$$C(x_0, \sigma(x_0))\sigma(x_0) = \delta_{cr}$$

The simultaneous solution of the joint algebraic system and condition (2.24) enables us (in the case of specified characteristics of the crack-resisting material) to determine the critical value of the external load (contact pressure), the size of the pre-fracture zone for the state of limit equilibrium and the temperature at which the crack appears.

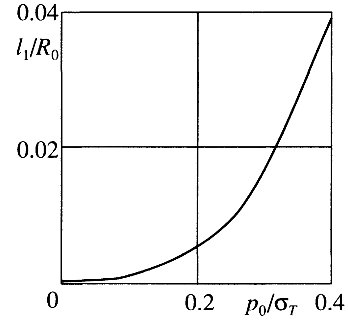

Fig. 4.

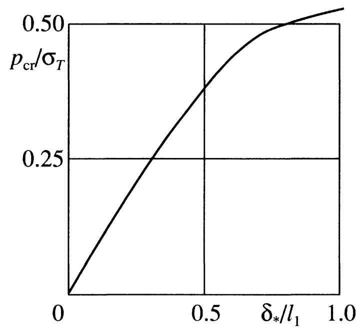

Fig. 5.

Graphs of the relative length of the pre-fracture zone against the dimensionless load p_0/σ_T and of the level of the critical load p_{cr}/σ_T against the relative opening δ_*/l at the centre of the zone, where $\delta_* = \pi\delta_{cr}G/[(1+k_0)\sigma_T]$ and σ_T is the tensile yield stress of the hub material, are presented in Figs. 4 and 5 for the hub of a borehole sucker rod pump with a speed of the plunger $V = 0.2$ m/s.

If the pre-fracture zone at one end reaches the inner surface of the hub, condition (2.15) becomes superfluous.

3. The case of an arbitrary number of pre-fracture zones

Now suppose there are N rectilinear pre-fracture zones of length $2l_k$ ($k = 1, 2, \ldots, N$) in the hub of a friction pair close to the friction surface during the operation process. At the centres of the zones, we place the origins of local systems of coordinates $x_kO_ky_k$, the axes x_k of which coincide with those of the zones and make angles α_k with the x axis (Fig. 1). In the case being investigated, the occurrence of crack nuclei represents the rupture of the material bonds between the surfaces of the pre-fracture zones of the hub. The dimensions of the zones are unknown in advance and must be determined when solving the problem.

The solution of the problem for this case is analogous to the solution in the case of a single pre-fracture zone. The complex potentials $\Phi_2(z)$, $\Psi_2(z)$ and $\Phi_3(z)$, $\Psi_3(z)$ are extended the case of an arbitrary number of zones. On satisfying the boundary conditions on the surfaces of the zones

$$\sigma_{y_k} - i\tau_{x_ky_k} = q_{y_k} - iq_{x_ky_k}, \quad k = 1, 2, \ldots, N \tag{3.1}$$

we obtain a system of N singular integral equations in the unknown functions $g_k(x_k)$ ($k = 1, 2, \ldots, N$). To this system it is necessary to add the conditions

$$\int_{-l_k}^{l_k} g_k(t)dt = 0, \quad k = 1, 2, \ldots, N \tag{3.2}$$

for the internal pre-fracture zones.

Using the procedure for converting a system to an algebraic system, the system of singular integral equations with conditions (3.2) is reduced to a system of $N \times M$ algebraic equations for determining the $N \times M$ unknowns $g_k(t_m)$ ($k = 1, 2, \ldots, N$; $m = 1, 2, \ldots, M$)

$$\frac{1}{M}\sum_{m=1}^{M}\sum_{k=1}^{N} l_k[g_k(t_m)R_{nk}(l_k t_m, l_n x_r) + \overline{g_k(t_m)}S_{nk}(l_k t_m, l_n x_r)] = f_n(x_r)$$
$$\sum_{m=1}^{M} g_n(t_m) = 0; \quad n = 1, 2, \ldots, N; \quad r = 1, 2, \ldots, M-1 \tag{3.3}$$

The functions $f_n(x_n)$ are defined by expression (2.14) when x_1, y_1 in them is replaced by x_n, y_n.

If we change to complex-conjugate quanties in Eq. (3.3), we obtain a further $N \times M$ algebraic equations. The required coefficients α_k, β_k of the contact pressure function and the unknown values of the normal q_{y_n} and shear $q_{x_n y_n}$ forces at the mesh points of the corresponding pre-fracture zone appear on the right-hand sides of the algebraic systems (3.3) in terms of the function $f_n(x_r)$ ($n = 1, 2, \ldots, N$) for the loading on the zone surfaces.

For the closure of the resulting algebraic systems, the basic relations of the problem must be supplemented by equations relating the displacement of the opening of zone surfaces with the corresponding bond tractions. Without loss of generality, these equations can be represented in the form

$$(\upsilon_k^+ - \upsilon_k^-) - i(u_k^+ - u_k^-) = C(x_k, \sigma_k)(q_{y_k}(x_k) - iq_{x_k y_k}(x_k)), \quad k = 1, 2, \ldots, N \tag{3.4}$$

where $\sigma_k = \sqrt{q_{y_k}^2 + q_{x_k y_k}^2}$ is the modulus of the vector of the tractions of the corresponding bonds of the zone; $C(x_k, \sigma_k)$ are the effective compliances of the corresponding bonds, which depend on the tension in the bonds.

Using the solution obtained, we find

$$g_k(x_k) = \frac{2G}{1+k_0}\frac{\partial}{\partial x_k}[C(x_k, \sigma_k)(q_{y_k}(x_k) - iq_{x_k y_k}(x_k))], \quad k = 1, 2, \ldots, N \tag{3.5}$$

where x_k is the affix of the points of the surfaces of the k-th pre-fracture zone.

These complex equations serve to determine the fractions q_{y_k} and $q_{x_k y_k}$($k = 1, 2, \ldots, N$ of the bonds between the corresponding zone surfaces.

In order to construct the missing algebraic equations for determining the approximate values of the fractions $q_{y_k}(t_m)$ and $q_{x_k y_k}(t_m)(m = 1, 2, \ldots, M)$ at the mesh points, we proceed in a similar manner as in the case of a single pre-fracture zone. As a result, a complex algebraic system consisting of $N \times M$ equations is obtained for determining the quantities $q_{y_k}(t_m), q_{x_k y_k}(t_m)(k = 1, 2, \ldots, N; m = 1, 2, \ldots, M)$ at the mesh points of the zones. The $2 \times N$ complex equations which determine the dimensions of the zones are not sufficient for the closure of the algebraic equations obtained.

The solution of the system of algebraic equations is sought in the class of functions (stresses) which are bounded everywhere. Consequently, it is necessary to add to system (3.3) the condition for the stresses at the ends of the pre-fracture zones $x_k = \pm l_k$ ($k = 1, 2, \ldots, N$) to the founded. These conditions differ from conditions (2.19) solely in the fact that the function $g_1(t_m)$ is replaced by the function $g_k(t_m)$.

The resolvents in $a_k, b_k, \alpha_k, \beta_k, g_k(t_m), q_{y_k}(t_m), q_{x_k y_k}(t_m)\,(k = 1, 2, \ldots, N; m = 1, 2, \ldots, M)$ which have been obtained enable us, in the case of a specified external load, to determine the stress-strain state of the hub of a contact pair when there is an arbitrary number of pre-fracture zones in the hub, the contact pressure, the temperature distribution and, also, the abrasive wear of the components of the friction pair.

The joint resolvent turns out to be non-linear even in the case of linear elastic bonds due to the unknown quantities θ_1, θ_1, l_k, ($k = 1, 2, \ldots, N$). The method of successive approximations is used to solve it.

To investigate the limit equilibrium when a crack occurs, we use relation (2.24) in which the modulus of the displacement vector on the pre-fracture zone surfaces is taken for the corresponding zone, that is, it is determined using formula (2.23) with the replacement of

$$l_1, A, B \text{ by } l_k, A_k = \sum_{m=1}^{M_1} \upsilon_k^0(t_m), \quad B_k = \sum_{m=1}^{M_1} u_k^0(t_m); \quad k = 1, 2, \ldots, N$$

The analysis of the model of the formation of a crack-type defect in the hub of a friction pair during operation reduces to the simultaneous parametric investigation of the resolvent of the contact problem, the system of algebraic Eqs. (3.3), (3.6), the finite-difference analogue of conditions (3.5) and the criterion for the appearance of a crack for different values of the free parameters of the frictional pair. These are the different thermophysical and mechanical characteristics of the materials, the geometrical dimensions of the hub and plunger and the rate of motion of the plunger.

At a certain stage of the loading, the simultaneous existence of pre-fracture zones in the hub of the contact pair and cracks which have been formed with bonds between the faces in the end zones is possible. The method of solution in this case involves simultaneously taking account of the pre-fracture zones and the cracks.

4. A bridged crack

Suppose there are N rectilinear cracks of length $2l_k$ ($k=1, 2, \ldots, N$) in an elastic hub close to the friction surface. Operational conditions of the friction pair, under which residual strains can occur, are assumed to be inadmissible. We place the origins of local systems of coordinates $x_kO_ky_k$ at the centres of the cracks and the O_kx_k axes of these local systems coincide with the lines of the cracks and make angles α_k with the Ox axis (Fig. 1). In the case being considered, it is necessary to assume that there are no bonds (ruptured bonds) in the middle part of the pre-fracture zones. In this case, a crack bonds with in an end zone of arbitrary size will be treated as a special case of a pre-fracture zone in the central part of which there are no cohesive forces (bonds) between the faces. It is assumed that the dimensions of the end zones of the crack are comparable with the crack length. The crack faces outside the end zones are free from external loads. We now separate out the parts of the cracks d_{1k} and d_{2k} (the end zones) adjacent to its tips where the crack faces interact.

During the operation of the friction pair, a normal force $q_{y_k}(x_k)$ and a shear force $q_{x_k y_k}(x_k)$ ($k=1, 2, \ldots, N$) will arise, in the general case, in the bonds connecting the crack faces under the action of the external force (the contact pressure and friction forces) and the thermal load on the hub. Consequently, normal and shear forces $q_{y_k}(x_k)$ and $q_{x_k y_k}(x_k)$ respectively will be applied to the crack faces in the end zones. These stresses are unknown in advance and are to be determined when solving the boundary-value problem in contact fracture mechanics. We recall that, in the case being considered, each crack consists of an internal domain, that is, the opposite faces of the crack, which are load-free, and end zones $(-l_k, \lambda_{1,k})$ and $(\lambda_{2,k}, l_k)$ with bonds between the faces.

In order to determine the contact pressure, it is again necessary to consider[6] the contact problem of the indentation of the plunger into the surface of the hub when there is wear.

To determine the displacements v_{1y}, it is necessary to solve the thermoelasticity problem (1.6)–(1.10) for a hub with cracks with the following conditions on the crack faces

$$\begin{aligned} &\sigma_n = 0, \ \tau_{nt} = 0 \ \text{ on the free face of the crack} \\ &\sigma_n = q_{y_k}(x_k), \quad \tau_{nt} = q_{x_k y_k}(x_k) \ \text{ on the surfaces of the end tones of the crack} \end{aligned} \tag{4.1}$$

The thermoelastic problem for the displacements of the contact surface of the plunger is formulated in a similar manner.

It is necessary to supplement the equations and conditions (1.1)–(1.12), (4.1) with relations connecting the opening of a crack to the bond tractions. Without loss of generality, we can represent these relations in the form of (1.13).

In order to solve the problem, it is necessary to solve the contact wear problem and fracture mechanics problem simultaneously. To determined the second stress-strain state in the hub, we have the boundary conditions (2.4) and (2.5) and, also, the conditions on the crack faces

$$\begin{aligned} &\bar{\bar{\sigma}}_{y_1} = -\bar{\sigma}_{y_1}, \ \bar{\bar{\tau}}_{x_1y_1} = -\bar{\tau}_{x_1y_1} \ \text{ on the free faces of the crack} \\ &\bar{\bar{\sigma}}_{y_1} = q_{y_1} - \bar{\sigma}_{y_1}, \quad \bar{\bar{\tau}}_{x_1y_1} = q_{x_1y_1} - \bar{\tau}_{x_1y_1} \ \text{ on the surfaces of the end tones of the crack} \end{aligned} \tag{4.2}$$

Using the Kolosov - Muskhelishvili formulae,[11] the boundary conditions of problem (2.4) and (2.5), (4.2) can be written in the form of a boundary-value problem for finding the complex potentials $\Phi(z)$ and $\Psi(z)$ for the hub.

We will seek the complex potentials in the form (2.7)–(2.10). On requiring that the complex potentials (2.7)–(2.10) should satisfy the boundary conditions of problem (2.4), (2.5), we obtain infinite systems of equations in the coefficients a_k and b_k. The coefficients of the expansion of the contact pressure (2.12) and integrals of the required function $g_1(t)$

occur on the right-hand sides of these systems. On satisfying boundary condition (4.2) on the crack faces with the functions (2.7)–(2.10), we obtain a singular integral equation of the form of (2.13) in the unknown function $g_1(x_1)$ with a transformed right-hand side

$$f_1(x_1) = \begin{cases} f_1^0(x_1) - (\bar{\sigma}_{y_1} - i\bar{\tau}_{x_1y_1}) & \text{on the free face of the crack} \\ f_1^0(x_1) + q_{y_1} - iq_{x_1y_1} - (\bar{\sigma}_{y_1} - i\bar{\tau}_{x_1y_1}) & \text{on the surfaces of the end tones of the crack} \end{cases} \quad (4.3)$$

The condition for the displacements to be unique on passing around the crack contour, which has a form analogous to (2.15), has to be added to the singular integral equation for an internal crack (see Fig. 1).

The algebraic system for finding the required coefficients α_k, β_k is constructed in the same way as in the preceding problem (see Section 2). As a result, we obtain an infinite algebraic system in α_k^0 $(k = 0, 1, 2, \ldots)$, β_k^0 $(k = 1, 2, \ldots)$ and α_k^1, β_k^1, etc. The unknown quantities θ_1 and θ_2 appear in this system in a non-linear form.

Using the procedure for conversion to an algebraic form, a singular integral equation of the form of (2.13), (4.3) with condition (2.15) is reduced to a system of M algebraic equations for determining the M unknowns $g_1(t_m)$ $(m = 1, 2, \ldots, M)$ of the form of (2.16). The unknown values of the normal force $q_{y_1}(x_r)$ and shear force $q_{x_1y_1}(x_r)$ at the mesh points of the end zones of the crack occur on the right-hand sides of system (2.16).

The supplementary relation (1.13) is the condition which determines the unknown tractions of the bonds between the faces of the end zones of a crack. In the problem in question, this additional condition is more conveniently written in the form of (2.17) for the derivative of the displacements of the crack faces in the end zones.

Eq. (2.18), where x_1 is the affix of the points of the crack faces in the end zones, can be written using the solution obtained. This complex equation serves to determine the tractions q_{y_1} and $q_{x_1y_1}$ in the bonds between the crack faces in the end zones.

In order to avoid solving of integrodifferential equations, we will represent the complex Eq. (2.18) in the form of the two real equations

$$-\frac{1+k_b}{2G}\int_{-l_1}^{x_1} v_1^0(x_1)dx_1 = C(x_1, \sigma_1)q_{y_1}(x_1), \quad -\frac{1+k_b}{2G}\int_{-l_1}^{x_1} u_1^0(x_1)dx_1 = C(x_1, \sigma_1)q_{x_1y_1}(x_1) \quad (4.4)$$

Here,

$$v_1^0(x_1) = v_1^+(x_1, 0) - v_1^-(x_1, 0), \quad u_1^0(x_1) = u_1^+(x_1, 0) - u_1^-(x_1, 0)$$

In order to construct the missing algebraic equations for determining the approximate values of the tractions $q_{y_1}(t_{1,m})$ and $q_{x_1y_1}(t_{1,m})$ at the mesh points, we require that condition (4.4) is satisfied at the mesh points $t_{1,m}$ $(m = 1, 2, \ldots, M_1)$ which are contained in the end zones of the crack. As a result, instead of each of Eq. (4.4), we obtain algebraic systems consisting of M_1 equations for determining the approximate values of $q_{y_1}(t_{1,m})$ and $q_{x_1y_1}(t_{1,m})$ $(m = 1, 2, \ldots, M_1)$ respectively

$$\begin{gathered} Av_1^0(t_{1,1}) = C(t_{1,1}, \sigma_1(t_{1,1}))q_{y_1}(t_{1,1}),\ A(v_1^0(t_{1,1}) + v_1^0(t_{1,2})) = C(t_{1,2}, \sigma_1(t_{1,2}))q_{y_1}(t_{1,2}), \ldots \\ \ldots,\ A\sum_{m=1}^{M_1} v_1^0(t_{1,m}) = C(t_{M_1}, \sigma_1(t_{M_1}))q_{y_1}(t_{M_1}) \end{gathered} \quad (4.5)$$

$$\begin{gathered} Au_1^0(t_{1,1}) = C(t_{1,1}, \sigma_1(t_{1,1}))q_{x_1y_1}(t_{1,1}),\ A(u_1^0(t_{1,1}) + u_1^0(t_{1,2})) = C(t_{1,2}, \sigma_1(t_{1,2}))q_{x_1y_1}(t_{1,2}), \ldots \\ \ldots, A\sum_{m=1}^{M_1} u_1^0(t_{1,m}) = C(t_{M_1}, \sigma_1(t_{M_1}))q_{x_1y_1}(t_{M_1}) \end{gathered} \quad (4.6)$$

where

$$A = -\frac{1+k_b}{2G}\frac{\pi l}{M}$$

and M is the number of mesh points belonging to the crack.

The two complex equations, determining the dimensions of the end zones, are insufficient to close the resulting algebraic equations.

Since the solution of integral Eq. (2.13) is sought in the class of functions (stresses) which are bounded everywhere, the conditions for the stresses to be bounded in the neighbourhood of the crack tips $x_1 = \pm l_1$ has to be added to system (2.17). These conditions have the form of (2.19).

The resulting algebraic system (2.16), (4.5), (4.6), (2.19) is non-linear because of the unknown dimensions of the end zones of the crack. The systems of equations in $a_k, b_k, \alpha_k, \beta_k, g_1(t_m) = v_1^0(t_m) - iu_1^0(t_m)\,(m = 1, 2, \ldots, M)$ and $q_{y_1}(t_{1,m}), q_{x_1y_1}(t_{1,m})\,(m = 1, 2, \ldots, M_1)$ enable one, in the case of a specified external load, to find the stress-strain state of the hub of the contact pair when there is a crack with bonds between the faces in the end zones in the hub, the contact pressure, the temperature distribution and, also, the abrasive wear of the components of the contact pair.

The simultaneous solution of the resulting systems of equations enables us to find approximate values of the coefficients $a_k, b_k, \alpha_k, \beta_k$, the values of the functions $v_1(t_m), u_1(t_m)\,(m = 1, 2, \ldots, M_1)$ and $q_{y_1}(t_{1,m}), q_{x_1y_1}(t_{1,m})\,(m = 1, 2, \ldots M_1)$ and the dimensions of the end zones of the crack.

The joint system of equations is non-linear even in the case of linear elastic bonds because of the unknown quantities $\theta_1, \theta_2, \lambda_{11}$ and λ_{21}. The method of successive approximations is used to solve it.

In each approximation, the joint algebraic system was solved using Gauss method with a choice of the main element.

In the case of a non-linear law of deformation of the bonds, an iterative algorithm, which is similar to the method of elastic solutions,[14] is also used to determine the tractions in the end zones.

Graphs of the length of the end zone of a crack $d_* = d_{21}/l_1$ for the hub of a borehole pump against the dimensionless value of the contact pressure p_0/σ_* is shown in Fig. 6 for different crack lengths $\lambda = l_1/(R_1 - R_0)$. The calculations were carried out for a hub as applied to a U8-6MA2 reciprocal borehole pump with a speed of the plunger of 0.2 m/s. The following parameters were taken as the constants: $2R_0 = 57$ mm, $2R_1 = 73$ mm, $2R'_0 = 56.7$ mm and the parameters (2.21).

The distributions of the normal forces q_{y_1}/p_0 in the bonds between the crack faces as a function of the dimensionless coordinate x_1/l_1 are shown in Fig. 7. Curve *1* corresponds to a linear bond and curve *2* to a bilinear bond, $d_{21}/l_1 = 0.3$.

The calculations show that, in the case of a linear law of deformation of the bonds, the bond tractions of always have a maximum value at the edge of an end zone. A similar pattern is also observed for the magnitudes of the openings of the cracks. In fact, the opening of a crack at the edge of an end zone has a maximum in the case of both linear and non-linear deformation laws and, moreover, the opening of a crack increases as the relative compliance of the bonds increases.

Using the solution of the problem, we will calculate the displacements on the faces of the left-hand end zone of the crack

$$-\frac{1+k_b}{2G}\int_{-l_1}^{x_1} g_1(x_1)dx_1 = v(x_1, 0) - iu(x_1, 0), \quad x_1 \in (-l_1, \lambda_{11})$$

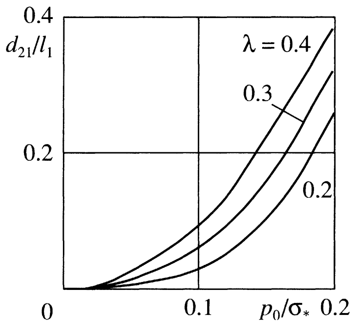

Fig. 6.

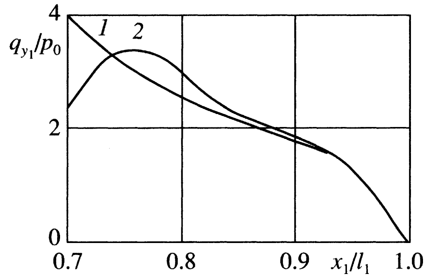

Fig. 7.

Applying this formula for the end zone of a crack when $x_1 = \lambda_{11}$, changing the variable and replacing the integral with a sum, we find

$$-\frac{1+k_b}{2G}\frac{\pi l_1}{M}\sum_{m=1}^{M_2} g_1(t_m) = \upsilon(\lambda_{11}, 0) - iu(\lambda_{11}, 0) \tag{4.7}$$

Here M_2 is the number of mesh points in the interval $(-l_1, \lambda_{11})$. Taking account of the fact that $g_1(t_m) = \upsilon_2^0(t_m) - iu_1^0(t_m)$, from equality (4.7) we find $\upsilon(\lambda_{11}, 0)$ and $u(\lambda_{11}, 0)$ and the modulus of the strain vector at the edge of the end zone of the crack when $x_1 = \lambda_{11}$

$$V_0^{(1)} = \sqrt{u^2+\upsilon^2} = \frac{1+k_b}{2G}\frac{\pi l_1}{M}\sqrt{A_{(1)}^2 + B_{(1)}^2}; \quad A_{(1)} = \sum_{m=1}^{M_2} \upsilon_1^0(t_m), \quad B_{(1)} = \sum_{m=1}^{M_2} u_1^0(t_m) \tag{4.8}$$

Similarly, for the modulus of the strain vector at the edge of the end zone of a crack when $x_1 = \lambda_{21}$, we have

$$V_0^{(2)} = \frac{1+k_b}{2G}\frac{\pi l_1}{M}\sqrt{A_{(2)}^2 + B_{(2)}^2}; \quad A_{(2)} = \sum_{m=M-M_1}^{M} \upsilon_1^0(t_m), \quad B_{(2)} = \sum_{m=M-M_1}^{M} u_1^0(t_m) \tag{4.9}$$

where M_1 is the number of mesh points in the interval (λ_{21}, l_1).

To determine the limit state at which the crack growth occurs, we use the critical condition

$$|(\upsilon^+ - \upsilon^-) - i(u^+ - u^-)| = \delta_{cr}$$

The equality

$$V_0^{(i)} = \delta_{cr} \quad i = 1, 2 \tag{4.10}$$

in conjunction with expressions (4.8) and (4.9) will then be the condition determining the limit value of the external load (the contact pressure).

Taking account of relation (1.13), the limit condition can also be written in the form

$$C(x_0, \sigma_1(x_0))\sigma_1(x_0) = \delta_{cr}$$

where $x_0 = \lambda_{11}$ for the left end zone and $x_0 = \lambda_{12}$ for the right end zone.

The simultaneous solution of the joint algebraic system and condition (4.1) enables us (in the case of specified characteristics of a crack-resistant material) to determine the critical value of the external load (the contact pressure), the dimensions of the end zones of a crack in the case of the limit equilibrium state, and the temperature at which the crack growth occurs.

The dependence of the critical load p_{cr}/σ_* on the dimensionless length of a crack $\lambda = l_1/(R_1 - R_0)$ in the hub of a borehole pump for a speed of the plunger $V = 0.2$ m/s is shown in Fig. 8.

If, at one end, the crack reaches the inner surface of the hub, the need for condition (2.15) becomes superfluous. In the case of a surface crack, condition (2.15) is replaced by the condition for the stresses at the edge, which reaches the surface $r = R_0$, to be finite.

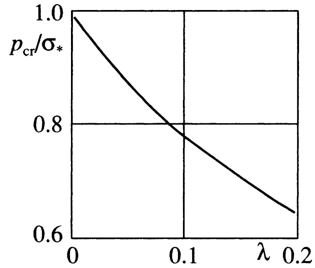

Fig. 8.

5. The case of an arbitrary number of bridged cracks

Now suppose there are N rectilinear bridged cracks of length $2l_k$ ($k = 1, 2, \ldots, N$) in the hub of a friction pair close to the friction surface during operation.

We shall assume that there are end zones in which the crack faces interact such that the interaction restrains the opening of the crack. It is assumed that such end zones are adjacent to the crack tips and that their dimensions can be comparable with the dimensions of the cracks. We denote by $L = L'_1 + \ldots + L'_N$, the set of free faces of the cracks and, by $L'' = L''_1 + \ldots + L''_N$, the set of end zones of the cracks in which the faces interact with the bonds.

The boundary conditions on the crack faces will have the form

$$\sigma_{y_k} - i\tau_{x_k y_k} = 0 \text{ in } L', \quad \sigma_{y_k} - i\tau_{x_k y_k} = q_{y_k} - iq_{x_k y_k} \text{ in } L'' \tag{5.1}$$

The solution of the problem for this case is analogous to the solution in the case of a single bridged crack. The complex potentials $\Phi_2(z)$, $\Psi_2(z)$ and $\Phi_3(z)$, $\Psi_3(z)$ are extended to the case of an arbitrary number of cracks with end zones. On satisfying the boundary conditions (5.1) on the crack faces with end zones, we obtain a system of N singular integral equations in the unknown functions $g_k(x_k)$ ($k = 1, 2, \ldots, N$). Conditions (3.2) have to be added to this system in the case of internal bridged cracks.

Using the procedure for converting to an algebraic system, the system of singular integral equations with conditions (3.2) mentioned above reduces to a system of $N \times N$ algebraic equations of the form of (3.3) for determining the $N \times M$ unknown $g_k(t_m)$ ($k = 1, 2, \ldots, N$; $m = 1, 2, \ldots, M$). The functions $f_n(x_n)$ are defined by expressions (4.3) when x_1, y_1 in them is replaced by x_n, y_n.

If we change to complex-conjugate quantities in system (3.3), we obtain a further $N \times M$ algebraic equations. The required coefficients α_k, β_k of the contact pressure function and the unknown values of the normal force q_{y_n} and shear force $q_{x_n y_n}$ at the mesh points of the end zones of the corresponding crack appear on the right-hand sides of the algebraic systems (3.3) in terms of the function $f_n(x_r)$ ($n = 1, 2, \ldots, N$) for the load on the crack faces. To close the algebraic system obtained, the main relations of the problem must be supplemented by Eq. (3.4) relating the displacements of the opening of the faces of the end zones and the corresponding bond tractions.

Using the solution obtained, we arrive at relations (3.5), where x_k is the affix of the points of the faces of the end zones of the k-th crack. These complex equations serve to determine the bond tractions q_{y_k} and $q_{x_k y_k}$ $(k = 1, 2, \ldots, N)$ in the end zones of the corresponding cracks.

In order to construct the missing algebraic equations for determining the approximate values of the tractions $q_{y_k}(t_{k,m})$ and $q_{x_k y_k}(t_{k,m})$ $(k = 1, 2, \ldots, N; m = 1, 2, \ldots, M_{1k})$ at the mesh points $t_{k,m}$ in the end zones, on proceeding in the same way as in the case of a single crack, we obtain a complex algebraic system consisting of $N \times M_{11}$ equations. The $2 \times N$ complex equations, determining the dimensions of the end zones, are insufficient to close the algebraic equations obtained.

The solution of the system of integral equations is sought in the class of functions (stresses) which are bounded everywhere. Consequently, it is necessary to add to Eq. (3.3) the conditions for the stresses at the crack tips $x_k = \pm l_k$ ($k = 1, 2, \ldots, N$) to be bounded. These conditions differ from conditions (2.19) solely in the fact that the function $g_1(t_m)$ is replaced by the function $g_k(t_m)$ ($k = 1, 2, \ldots, N$).

The resolvents in a_k, b_k, α_k, β_k, $g_k(t_m)$ ($k = 1, 2, \ldots, N$; $m = 1, 2, \ldots, M$) and $q_{y_k}(t_{k,m}), q_{x_k y_k}(t_{k,m})$ $(k = 1, 2, \ldots, N; m = 1, 2, \ldots, M_{1k})$ which are obtained enable us, in the case of a specified external load, to determine the stress-strain state of the hub of a contact pair when there is an arbitrary number of cracks with end zones in the hub, the contact pressure, the temperature distribution and, also, the abrasive wear of the components of the friction pair.

The joint resolvent turns out to be non-linear even in the case of linear elastic bonds due to the unknown quantities θ_1, θ_2, λ_{1k} and λ_{2k} ($k = 1, 2, \ldots, N$). The method of successive approximations is used to solve it.

To investigate the limit equilibrium when a crack grows, we use relation (4.10) in which the modulus of the strain vector at the edge of the end zone is taken for the corresponding crack, that is, it is determined using formulae (4.8) and (4.9) when l_1, $A_{(i)}$, $B_{(i)}$ is replaced by l_k, $A_k^{(j)}$, $B_k^{(j)}$ ($j = 1, 2$) and

$$A_k^{(1)} = \sum_{m=1}^{M_{2k}} \upsilon_k^0(t_{k,m}),\ B_k^{(1)} = \sum_{m=1}^{M_{2k}} u_k^0(t_{k,m}),\ A_k^{(2)} = \sum_{m=M-M_{1k}}^{M} \upsilon_k^0(t_{k,m}),\ B_k^{(2)} = \sum_{m=M-M_{1k}}^{M} u_k^0(t_{k,m})$$

We recall that, here, M_{2k} is the number of mesh points in the end zone ($-l_k$, λ_{1k}) and M_{1k} is the number of mesh points in the interval (λ_{2k}, l_k).

The analysis of the model of a bridged crack in the hub of a friction pair during operation reduces to a simultaneous parametric investigation of the resolvent of the contact problem, the system of algebraic Eq. (3.3) with the conditions that the stresses should be finite (an analogue of conditions (2.19)), the finite-difference analogue of conditions (3.5) and the criterion for the crack growth for different values of the free parameters of the friction pair, that is, the thermophysical and mechanical characteristics of the materials, the geometrical dimensions of the hub and the plunger, and the speed of the plunger.

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